Examples of one-parameter automorphism groups of UHF algebras

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1 Introduction

B. Blackadar [1] constructed for the first time an example of a symmetry (or an automorphism of period two) of the CAR algebra (or the UHF algebra of type 2^∞) whose fixed point algebra is not AF (or approximately finite-dimensional). This was soon extended to produce an example of finite-group actions on UHF algebras whose fixed point algebras are not AF [5] and then of compact-group actions [12]. Note that these examples can now be obtained as corollaries [13, 4] to the classification results for certain amenable C^*-algebras started by G.A. Elliott [11] and extended by himself and many others (see e.g. [10]).

In the same spirit we present yet other examples, this time, of one-parameter automorphism groups of UHF algebras, which do not seem to follow as a consequence from the above general results. Before stating what kind of examples they are we first recall the subject from [8, 3, 17].

Let A be a UHF algebra (or more generally, a simple AF C^*-algebra) and let α be a one-parameter automorphism group of A. We always assume that $t \mapsto \alpha_t(x)$ is continuous for each $x \in A$ and denote by $\delta = \delta_\alpha$ the (infinitesimal) generator of α:

$$\delta(x) = \lim_{t \longrightarrow 0} (\alpha_t(x) - x)/t, \quad x \in \mathcal{D}(\delta),$$

where the domain $\mathcal{D}(\delta)$ of δ is the set of $x \in A$ for which the limit exists. Then $\mathcal{D}(\delta)$ is a dense *-subalgebra of A and δ is a *-derivation of $\mathcal{D}(\delta)$ into A. We equip $\mathcal{D}(\delta)$ with the norm $\|\cdot\|_\delta$:

$$\|x\|_\delta = \| \begin{pmatrix} x & \delta(x) \\ 0 & x \end{pmatrix} \|_{M_2 \otimes A},$$

by which $\mathcal{D}(\delta)$ is a Banach *-algebra. We note that the C^*-algebra A can be recovered as the universal C^*-algebra of $\mathcal{D}(\delta)$ and that the self-adjoint part of $\mathcal{D}(\delta)$ is closed under C^∞ functional calculus. Let us say that a Banach *-algebra (or a C^*-algebra) D is AF if D has an increasing sequence (D_n) of finite-dimensional *-subalgebras such that the union $\cup_n D_n$ is dense in D.

If α_t is given as $\mathrm{Ad}\, e^{ith}$ with some $h = h^* \in A$, then α is called inner and the generator δ_α is defined on the whole of A and is given by $\mathrm{ad}\, ih$. If α is obtained as the limit (pointwise on A and uniformly on compact subsets of $\mathbf{R}$) of inner one-parameter automorphism groups, then α is called approximately inner.

For a one-parameter automorphism group α of the UHF algebra A with generator δ_α we quote the following two results of S. Sakai [16, 17]:

Theorem 1.1 *The Banach *-algebra $\mathcal{D}(\delta_\alpha)$ contains an AF Banach *-subalgebra B such that B is dense in A under the embedding $\mathcal{D}(\delta_\alpha) \subset A$.*

Theorem 1.2 *If $\mathcal{D}(\delta_\alpha)$ is AF, then α is approximately inner.*

The condition of Theorem 1.2 was satisfied for all the known examples so far and the core problem ([17], 4.5.10), as a possible solution to the Powers-Sakai conjecture ([17], 4.5.9), asks whether this is true for all one-parameter automorphism groups of UHF algebras. We will give an example which shows this is not the case; we construct an approximately inner one-parameter automorphism group α such that $\mathcal{D}(\delta_\alpha)$ is not AF (see 2.1). The property we use to conclude this is real rank [9].

So far we know of no examples of one-parameter automorphism groups α of UHF algebras such that $\mathcal{D}(\delta_\alpha)$ contains no maximal abelian C^*-subalgebra (masa) of A. But we will present another example which shows that $\mathcal{D}(\delta_\alpha)$ need not contain a canonical AF masa of A even though $\mathcal{D}(\delta_\alpha)$ is AF (see 3.4). (We call an abelian C^*-subalgebra C of A a canonical AF masa if there exists an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that $A = \overline{\cup_n A_n}$ and $C \cap A_n \cap A'_{n-1}$ is a masa of $A_n \cap A'_{n-1}$ for all n with $A_0 = 0$.) As will be shown in 3.1, this is equivalent to the property that any inner perturbation of α is not AF locally representable. (We call α *AF locally representable* if there exists an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A with dense union such that α leaves each A_n invariant and thus $\alpha|A_n$ is inner. In this case there is a canonical AF masa C associated with (A_n) such that $\delta_\alpha|C = 0$, and the union $\cup_n A_n$ is a core for the generator δ_α and thus $\mathcal{D}(\delta_\alpha)$ is AF.) In our example, if α is periodic, we use the property that the fixed point algebra A^α is not AF to conclude that $D(\delta_\alpha)$ contains no canonical AF masa. If α is not periodic, we instead use the property that for some unitary u in $\mathcal{D}(\delta_\alpha)$ with $\delta_\alpha(u) \approx 0$ there is no continuous path (u_t) of unitaries between u and 1 such that $\delta_\alpha(u_t) \approx 0$.

To sum up let us state three properties for α:

(1) $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa.

(2) $\mathcal{D}(\delta_\alpha)$ is AF.

(3) α is approximately inner.

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) but (1) $\not\Leftarrow$ (2) $\not\Leftarrow$ (3).

2 A generator whose domain is not AF

Theorem 2.1 *Let A be a non type I simple AF algebra. Then there exists an approximately inner one-parameter automorphism group α of A such that the domain $\mathcal{D}(\delta_\alpha)$ is not AF.*

Proof. Let (A_n) be an increasing sequence of finite-dimensional C^*-subalgebras of A such that $A = \overline{\cup_n A_n}$ and let $A_n = \oplus_{j=1}^{k_n} A_{nj}$ be the direct sum decomposition of A_n into full matrix algebras A_{nj}. Since $K_0(A_n) \cong \mathbf{Z}^{k_n}$, we obtain a sequence of K_0 groups:

$$\mathbf{Z}^{k_1} \stackrel{\chi_1}{\to} \mathbf{Z}^{k_2} \stackrel{\chi_2}{\to} \cdots,$$

where χ_n is the positive map of $K_0(A_n) = \mathbf{Z}^{k_n}$ into $A_{n+1} = \mathbf{Z}^{k_{n+1}}$ induced by the embedding $A_n \subset A_{n+1}$. Since $K_0(A)$ is a simple dimension group other than $\mathbf{Z}$, we may assume that all $\chi_n(i,j) \geq 3$.

By using (A_n) we will express A as an inductive limit of C^*-algebras $A_n \otimes C[0,1]$. First we define a homomorphism $\varphi_{n,ij}$ of $A_{nj} \otimes C[0,1]$ into $A_{nj} \otimes M_{\chi_n(i,j)} \otimes C[0,1]$, with M_k the full k by k matrix algebra, as follows:

$$\varphi_{n,ij}(x)(t) = x(t) \oplus \oplus_{\ell=0}^{\chi_n(i,j)-2} x(\frac{t+\ell}{\chi_n(i,j)-1}),$$

in particular $\varphi_{n,ij}(x)$ is of diagonal form in the matrix algebra over $A_{nj} \otimes C[0,1]$. (In the above definition of $\varphi_{n,ij}$, the variable t inside the summands after the first could be removed if we assumed that $\min_{ij} \chi_n(i,j) \to \infty$.) Then on the embedding of

$$\oplus_{j=1}^{k_n} A_{nj} \otimes M_{\chi_n(i,j)} \otimes C[0,1]$$

in the natural way into $A_{n+1,i} \otimes C[0,1]$, $(\varphi_{n,ij})$ defines an injective homomorphism $\varphi_n : A_n \otimes C[0,1] \to A_{n+1} \otimes C[0,1]$. Then it follows that the inductive limit of the sequence $(A_n \otimes C[0,1], \varphi_n)$ is isomorphic (as we shall explain) to the given C^*-algebra A; we have thus expressed A as $\overline{\cup_n B_n}$ where $B_n = A_n \otimes C[0,1] \subset B_{n+1}$.

This isomorphism follows from Elliott's result [11] by checking that the inductive limit C^*-algebra is simple and of real rank zero [2] and has the right K-theoretic data, these properties being consequences of the condition $\chi_n(i,j) \geq 3$ and the special form of $\varphi_{n,ij}$. (As a matter of fact, the proof that the inductive limit is AF is about the same as the proof that it has real rank zero, since both of these facts follow by showing that the canonical self-adjoint element $x_n \in 1 \otimes C[0,1] \subset B_n$, see below, can be approximated by self-adjoint elements with finite spectra in $A \cap (A_n \otimes 1)'$.)

We will define a one-parameter automorphism group α of A with the property $\alpha_t(B_n) = B_n$. First we define a sequence (H_n) with self-adjoint $H_n \in A_n \otimes 1 \subset B_n$ inductively. Let $H_1 \in A_1 \otimes 1 \subset B_1$ and let $H_n = H_{n-1} + \sum_i \sum_j h_{n,ij}$, where $h_{n,ij}$ is a self-adjoint diagonal matrix of $M_{\chi_{n-1}(i,j)}$ which is identified with a C^*-subalgebra of B_n by

$$M_{\chi_{n-1}(i,j)} \equiv 1 \otimes M_{\chi_{n-1}(i,j)} \otimes 1 \subset A_{n-1,j} \otimes M_{\chi_{n-1}(i,j)} \otimes 1 \subset B_n.$$

We define $\alpha_t|B_n$ by $\mathrm{Ad}\, e^{itH_n}|B_n$. Since $\alpha_t|B_n = \mathrm{Ad}\, e^{itH_{n+1}}|B_n$ from the definition of H_{n+1}, $(\alpha_t|B_n)$ defines a one-parameter automorphism group α of A. Let x be the identity function on the interval $[0,1]$ and let $x_n = 1 \otimes x \in 1 \otimes C[0,1] \subset B_n$. Then we have that $\alpha_t(x_n) = x_n,\ t \in \mathbf{R}$, or $\delta(x_n) = 0$ for the generator δ of α.

Note that $\mathcal{D}(\delta)$ contains $\cup_n B_n$. Since $(1 \pm \delta) \cup_n B_n = \cup_n B_n$ and $\|(1 \pm \delta)x\| \geq \|x\|$ for $x \in \mathcal{D}(\delta)$, it follows that $\cup_n B_n$ is a core for δ, i.e., $\cup_n B_n$ is dense in the Banach *-algebra $\mathcal{D}(\delta)$. See [8, 17] for details.

If $\mathcal{D}(\delta)$ is AF, then for any $h \in \mathcal{D}(\delta)_{sa} = \{y \in D(\delta)\,;\, y = y^*\}$ there exists a sequence (h_n) in $\mathcal{D}(\delta)_{sa}$ such that $\mathrm{Sp}(h_n)$ is finite and $\|h - h_n\|_\delta \to 0$. (We can further impose, without difficulty, the condition on (h_n) that $\mathrm{Sp}(h_n)$ is a subset of the smallest closed interval containing $\mathrm{Sp}(h)$.) Here we note that the spectrum of h_n, $\mathrm{Sp}(h_n)$, may be computed in $\mathcal{D}(\delta)$ or in A since they are the same. (If $y \in \mathcal{D}(\delta)$ is invertible in A, then it follows that $y^{-1} \in \mathcal{D}(\delta)$ and $\delta(y^{-1}) = -y^{-1}\delta(y)y^{-1}$.) In this case we may say that the Banach *-algebra $\mathcal{D}(\delta)$ has real rank zero as in the case of C^*-algebras [9]. What we will do is show that $\mathcal{D}(\delta)$ does not have real rank zero for certain α (and hence has real rank one, by defining real rank for $\mathcal{D}(\delta)$ as in [9]).

We fix H_1 and all $h_{n,ij}$ except for $h_{n,11}$. We will inductively define $h_{n,11}$ to be of the form

$$a_n \oplus 0 \oplus \cdots \oplus 0 \in 1 \otimes M_{\chi_{n-1}(1,1)} \otimes 1 \subset A_{n1} \otimes C[0,1]$$

with $a_n > 0$, to make sure that no x_n can be approximated by self-adjoint elements with finite spectra in $\mathcal{D}(\delta)$.

Let P_n be the identity of $B_{n1} = A_{n1} \otimes C[0,1]$ and let Q_n be the projection

$$1 \oplus 0 \oplus \cdots \oplus 0 \in 1 \otimes M_{\chi_{n-1}(1,1)} \otimes 1 \subset B_{n1}.$$

Let (ϵ_m) be a strictly decreasing sequence of positive numbers such that $\epsilon_1 \leq 3/5$. We shall construct a sequence (a_n) such that if $h = h^* \in B_{m+1,1}$ satisfies that $0 \leq h \leq 1$, $\mu(\mathrm{Sp}(h)) < \epsilon_{m+1}$, and $\|h - x_n P_{m+1}\| < 1/5$ for some $n \leq m$, then $\|\delta(h)\| > 1$, where μ denotes Lebesgue measure on $\mathbf{R}$. (Here we have imposed the condition $0 \leq h \leq 1$, which does not cause any loss of generality.) This in particular shows that if $h = h^* \in A$ with $0 \leq h \leq 1$ belongs to $\cup_n B_n$ and satisfies that $\mathrm{Sp}(h)$ is finite and $\|x_n - h\| < 1/5$ for some n, then $\|\delta(h)\| > 1$. We will discuss later how to remove the condition $h \in \cup_n B_n$ in this statement.

Let $m = 1$ and choose a_1 arbitrarily. If $h \in B_{2,1}$ with $0 \leq h \leq 1$, $\mu(\mathrm{Sp}(h)) < \epsilon_2$, and $\|\delta(h)\| \leq 1$, then

$$\begin{aligned} \|\delta(h)\| &\geq \|Q_2[iH_1 + \sum_i \sum_j ih_{2,ij}, h](1 - Q_2)\| \\ &\geq a_2\|Q_2 h(1 - Q_2)\| - 2\|H_1\| - \sum_{j=2}^{k_1} \|h_{2,1j}\| \end{aligned}$$

since $Q_2 h_{2,11} = a_2 Q_2$ and $Q_2 h_{2,ij} = 0$ for $(i,j) \neq (1,1)$. If a_2 is sufficiently large, then

$$\|Q_2 h(1 - Q_2)\| < \frac{\epsilon_1 - \epsilon_2}{4\xi_2},$$

where ξ_n is defined by $A_{n,1} \cong M_{\xi_n}$. If $\tilde{h} = Q_2 h Q_2 + (1-Q_2)h(1-Q_2)$, then we have that

$$\|h - \tilde{h}\| < \frac{\epsilon_1 - \epsilon_2}{2\xi_2}.$$

Since $h \in A_{2,1} \otimes C[0,1]$, there are at most ξ_2 eigenvalues of $h(t)$ for each $t \in [0,1]$. Thus $\mathrm{Sp}(h)$ consists of at most ξ_2 closed intervals. Since

$$\mathrm{Sp}(\tilde{h}) \subset \mathrm{Sp}(h) + [-\|h-\tilde{h}\|, \|h-\tilde{h}\|],$$

we obtain that

$$\mu(\mathrm{Sp}(\tilde{h})) < \epsilon_2 + 2\xi_2\|h - \tilde{h}\| < \epsilon_1 \leq 3/5,$$

which, in particular, implies that $\mu(\mathrm{Sp}(Q_2 h Q_2)) < 3/5$. On the other hand, if $\|x_1 P_2 - h\| < 1/5$, then $\|x_1 Q_2 - Q_2 h Q_2\| < 1/5$, which implies that

$$\begin{aligned} \|Q_2 h(0) Q_2\| &< 1/5 \\ \|Q_2 - Q_2 h(1) Q_2\| &< 1/5. \end{aligned}$$

Since $t \mapsto \mathrm{Sp}(Q_2 h(t) Q_2)$ is continuous in a certain well-known sense, it follows that $\mathrm{Sp}(Q_2 h Q_2)$ strictly contains $[1/5, 4/5]$, and so $\mu(\mathrm{Sp}(Q_2 h Q_2)) > 3/5$. This contradiction completes the proof for the case $m = 1$.

Suppose that $a_1, \ldots, a_m$ are chosen in such a way that the conditions for $h \in B_{n+1,1}$ with $n < m$ are satisfied. Let $h \in B_{m+1,1}$ be such that $0 \leq h \leq 1$, $\mu(\mathrm{Sp}(h)) < \epsilon_{m+1}$, and $\|\delta(h)\| \leq 1$. Then as before we have that

$$\begin{aligned} \|\delta(h)\| &\geq \|Q_{m+1}\delta(h)(1-Q_{m+1})\| \\ &\geq a_{m+1}\|Q_{m+1}h(1-Q_{m+1})\| - 2\|H_m\| - \sum_{j=2}^{k_{m+1}} \|h_{m+1,1j}\|. \end{aligned}$$

We choose a_{m+1} in such a way that $\|\delta(h)\| \leq 1$ implies that

$$\|Q_{m+1}h(1-Q_{m+1})\| < \frac{\epsilon_m - \epsilon_{m+1}}{4\xi_{m+1}}.$$

Then for $\tilde{h} = Q_{m+1} h Q_{m+1} + (1-Q_{m+1})h(1-Q_{m+1})$ we have that

$$\|h - \tilde{h}\| < \frac{\epsilon_m - \epsilon_{m+1}}{2\xi_{m+1}}.$$

Since $\mathrm{Sp}(h)$ consists of at most ξ_{m+1} connected components and $\mu(\mathrm{Sp}(h)) < \epsilon_{m+1}$, it follows that $\mu(\mathrm{Sp}(\tilde{h})) < \epsilon_m$. If $\|x_n P_{m+1} - h\| < 1/5$ for some $n < m+1$, then

$$\|Q_{m+1}x_n - Q_{m+1}hQ_{m+1}\| < 1/5.$$

If $n = m$, then by using that $Q_{m+1}x_m(0) = 0$ and $Q_{m+1}x_m(1) = Q_{m+1}$ we can reach a contradiction as before. If $n < m$, then since φ_m restricts to an isomorphism of $B_{m1} =$

$A_{m1}\otimes C[0,1]$ onto $Q_{m+1}B_{m+1,1}Q_{m+1}$ mapping x_nP_m into x_nQ_{m+1}, the pre-image $k\in B_{m1}$ of $Q_{m+1}hQ_{m+1}$ satisfies that $0\leq k\leq 1$, $\mu(\mathrm{Sp}(k))<\epsilon_m$, $\|\delta(k)\|\leq 1$, and $\|x_nP_m-k\|<1/5$. Thus this would be a contradiction by the induction hypothesis.

Thus we have shown that if $h=h^*\in\cup_nB_n$ has finite spectrum in $[0,1]$ and $\|h-x_n\|<1/5$ for some n then $\|\delta(h)\|>1$.

Let $h=h^*\in\mathcal{D}(\delta)$ be such that $\mathrm{Sp}(h)$ is a finite subset of $[0,1]$. Since $\cup_nB_n$ is a core for δ (or $\cup_nB_n$ is dense in $\mathcal{D}(\delta)$), we obtain a sequence (h_n) in $\cup_nB_n$ such that $\|h-h_n\|\to 0$ and $\|\delta(h)-\delta(h_n)\|\to 0$. Obviously we may suppose that $h_n=h_n^*$. If $\mathrm{Sp}(h)=\{\lambda_1,\lambda_2,\ldots,\lambda_k\}$ and $\epsilon>0$ is smaller than any $|\lambda_i-\lambda_j|/2$ with any $i\neq j$, $\mathrm{Sp}(h_n)$ is covered by the disjoint union of the ϵ-neighborhoods of $\lambda_1,\lambda_2,\ldots,\lambda_k$ for all sufficiently large n. If we define

$$p_{ni}=\frac{1}{2\pi i}\int_{|z-\lambda_i|=\epsilon}(z-h_n)^{-1}dz,$$

which is a projection in $\cup_nB_n$, we have that $\|h_n-\sum_i\lambda_ip_{ni}\|\approx 0$, depending on $\|h_n-h\|\approx 0$. If we define a projection p_i just as p_{ni} by using h instead of h_n, we obtain that $p_i\in\mathcal{D}(\delta)$, $h=\sum_i\lambda_ip_i$, $\|p_{ni}-p_i\|\to 0$, and $\|\delta(p_{ni})-\delta(p_i)\|\to 0$. (Here the latter convergence follows by using $\delta((z-h_n)^{-1})=(z-h_n)^{-1}\delta(h_n)(z-h_n)^{-1}$.) Hence it follows that $\tilde{h}_n=\sum_i\lambda_ip_{ni}\in\cup_nB_n$ satisfies that $\mathrm{Sp}(\tilde{h}_n)=\mathrm{Sp}(h)$, $\|\tilde{h}_n-h\|\to 0$, and $\|\delta(\tilde{h}_n)-\delta(h)\|\to 0$. In this way if furthermore $\|h-x_n\|<1/5$ for some n, we can conclude that $\|\delta(h)\|>1$. (Here to get the strict inequality instead of $\|\delta(h)\|\geq 1$, we may apply this argument to λh with $0<\lambda<1$ and $\|\lambda h-x_n\|<1/5$ instead of the given h.)

In the situation of the above proof we define a masa (maximal abelian C^*-subalgebra) C_n of A_n with $H_n\in C_n$ as follows. Let C_1 be a masa of A_1 containing H_1. We inductively define a masa C_n of A_n as the C^*-subalgebra generated by C_{n-1} and a masa $A_n\cap A'_{n-1}$ containing $h_{n-1,ij}$ for all i,j. Then $C=\overline{\cup_nC_n}$ is a maximal abelian AF C^*-subalgebra of $\tilde{A}=\overline{\cup_nA_n}$, which is regarded as a C^*-subalgebra of $A=\overline{\cup_nB_n}$ and is isomorphic to A itself. Let D_n be the C^*-subalgebra of $B_n=A_n\otimes C[0,1]$ generated by C_n and $1\otimes C[0,1]$. Then (D_n) forms an increasing sequence and generates a masa D of A. Since our generator δ vanishes on D, $\mathcal{D}(\delta)$ contains the C^*-algebra D. Furthermore D is Cartan in the sense that the unitary subgroup $\{u\in A\,;\,uDu^*=D\}$ generates A while C is Cartan in $\tilde{A}$. Since α fixes $\tilde{A}$, we may consider the one-parameter automorphism group $\tilde{\alpha}=\alpha|\tilde{A}$, whose generator $\tilde{\delta}$ vanishes on C. Since α (resp. $\tilde{\alpha}$) fixes the generating sequence (B_n) of A (resp. (A_n) of $\tilde{A}$) and is inner on each B_n (resp. A_n), both α and $\tilde{\alpha}$ can be called *locally representable* (or locally inner). (To be more specific, we call a one-parameter automorphism group α of a C^*-algebra A locally representable if there is an increasing sequence (A_n) of C^*-subalgebras of A such that α leaves A_n invariant and $\alpha|A_n$ is inner for each n and the union $\cup_nA_n$ is dense in A.) We will call $\tilde{\alpha}$ especially *AF locally representable* (or AF representable) since the A_n's are finite-dimensional. (What we meant by locally representable in [14] is in this latter sense.) Thus α does not look very queer as $\tilde{\alpha}$ does not, which may make the following problem interesting:

Problem For a unital simple AF C^*-algebra A is there a one-parameter automorphism group α of A such that α is not an inner perturbation of a locally representable one?

As a matter of fact this is probably what is meant by [17], 4.5.8. A locally representable one-parameter automorphism group α might be characterized, up to inner perturbation, by the property that $\mathcal{D}(\delta_\alpha)$ contains a masa of A. (In any case a similar problem is to find α such that $\mathcal{D}(\delta_\alpha)$ contains no masa.) Though this looks optimistic, we will consider a special case in the next section.

We shall conclude this section with a remark on *commutative normal *-derivations* introduced by S. Sakai [17].

A commutative normal *-derivation δ in an AF C^*-algebra A is defined as follows: δ is defined on $\mathcal{D}(\delta) = \cup_n A_n$ for some increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A with dense union and has a mutually commuting family $\{h_n\}$ of self-adjoint elements in A such that $\delta|A_n = \mathrm{ad}\, ih_n|A_n$. (This is adapted from 4.1.5 and 4.5.7 in [17] to the case of AF C^*-algebras.) Then it follows that δ extends to a generator, which generates an approximately inner one-parameter automorphism group (see [17], 4.1.11, and [7] for a similar result).

Remark 2.2 If A is a non type I simple AF C^*-algebra, there is a commutative normal *-derivation whose closure is not a generator.

Such an example is given in the proof of Theorem 2.1. The proof that $\overline{\cup_n B_n}$ with $B_n = A_n \otimes C[0,1]$ is AF (or even just the fact that this C^*-algebra is AF) shows that any finite subset in $\cup_n B_n$ can be approximately contained in a finite-dimensional C^*-subalgebra of B_m for some m. Hence we can construct an increasing sequence (D_n) of finite-dimensional C^*-subalgebras in $\cup_n B_n$ such that $\overline{\cup_n D_n} = \overline{\cup_n B_n}$. Thus there is a subsequence (k_n) such that $D_n \subset B_{k_n}$, which shows that $\delta|D_n = \mathrm{ad}\, iH_{k_n}|D_n$. This way $\delta_0 = \delta| \cup_n D_n$ is a commutative *-derivation. If the closure $\overline{\delta_0}$ were a generator, then it must be δ and $\mathcal{D}(\overline{\delta_0})$ would be AF, which is a contradiction.

3 AF locally representable actions

When A is a unital simple AF C^*-algebra and C is a maximal abelian AF C^*-subalgebra of A, we call C a canonical AF masa if there is an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that $\cup_n A_n$ is dense in A and $C \cap A_n \cap A'_{n-1}$ is a masa of $A_n \cap A'_{n-1}$ with $A_0 = 0$. Hence C is generated by $C_n = C \cap A_n \cap A'_{n-1}$, $n = 1, 2, \ldots$; there is a natural homomorphism from the infinite tensor product $\otimes_{n=1}^\infty C_n$ onto C. (See [18] for this kind of masa.) Then we note the following:

Proposition 3.1 *Let α be a one-parameter automorphism group of a unital simple AF C^*-algebra A. Then the following conditions are equivalent:*

1. *$\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa of A.*

2. *There is an $h = h^* \in A$ and an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A such that $\cup_n A_n$ is dense in $\mathcal{D}(\delta_\alpha) = \mathcal{D}(\delta_\alpha + \mathrm{ad}\, ih)$ and $\delta_\alpha +$*

$\mathrm{ad}\, ih$ *leaves* A_n *invariant, i.e., an inner perturbation of* δ_α *generates an AF locally representable one-parameter automorphism group of* A.

We only have to show (1) implies (2). When C denotes the canonical masa contained in $\mathcal{D}(\delta)$ with $\delta = \delta_\alpha$ and (A_n) denotes the associated increasing sequence, the proof will go as follows. We first find a self-adjoint $h \in A$ such that $\delta|C = -\mathrm{ad}\, ih|C$. Then $\delta + \mathrm{ad}\, ih$ vanishes on C and we may take $\delta + \mathrm{ad}\, ih$ for δ and assume that $\delta|C = 0$. We then modify (A_n) by employing a method in [16] in such a way that $\cup_n A_n \subset \mathcal{D}(\delta)$. Next we find a self-adjoint $h_n \in C$ such that $\delta|A_n = \mathrm{ad}\, ih_n|A_n$. Using the fact that h_n can be chosen from C we find a self-adjoint $h \in C$ such that $\delta + \mathrm{ad}\, ih$ leaves $\cup_n A_n$ invariant [17]. Thus by taking $\delta + \mathrm{ad}\, ih$ for δ and passing to a subsequence of (A_n), we may assume that $C \cup (\cup_n A_n) \subset \mathcal{D}(\delta)$, $\delta|C = 0$, and $\delta(A_n) \subset A_{n+1}$. Then we find a self-adjoint $h_n \in C \cap A_{n+1}$ such that $\delta|A_n = \mathrm{ad}\, ih_n|A_n$. If we denote by B_n the C^*-subalgebra generated by A_n and $C \cap A_{n+1}$, then (B_n) is an increasing sequence of finite-dimensional C^*-subalgebras of A with $\overline{\cup_n B_n} = A$ and $\delta(B_n) \subset B_n$ for all n. Thus this will complete the proof.

In the above argument most of the steps are either straightforward or given in [17]. An exception may be the assertion made in the very beginning, which we shall show below.

Lemma 3.2 *There exists an* $h = h^* \in A$ *such that* $\delta(x) = \mathrm{ad}\, ih(x),\ x \in C$.

Proof. Since $\mathcal{D}(\delta) \supset C$ and C is a C^*-algebra, $\delta|C$ is bounded [15]. We first show that

$$\|\delta|C \cap A_n'\| \to 0.$$

Suppose, on the contrary, that there is an $\epsilon > 0$ such that $\|\delta|C \cap A_n'\| > \epsilon$ for all n. Since the closure of the convex hull of the projections $\mathcal{P}(C \cap A_n')$ in $C \cap A_n'$ equals $\{h \in C \cap A_n' \,;\, 0 \leq h \leq 1\}$, we may assume that there is an $\epsilon > 0$ such that $\|\delta|\mathcal{P}(C \cap A_n')\| > \epsilon$ for all n. Thus we can find a sequence (e_n) of projections with $e_n \in C \cap A_n'$ such that $\|\delta(e_n)\| > \epsilon$. Since $\delta(e_n) = e_n\delta(e_n)(1-e_n) + (1-e_n)\delta(e_n)e_n$ and $\|e_n\delta(e_n)(1-e_n)\| > \epsilon$, there exists a state φ_n of A for any $\gamma \in (0,1)$ such that

$$\varphi_n(e_n) = 1 - \gamma, \quad \varphi_n\delta(e_n) > 2\epsilon\sqrt{\gamma(1-\gamma)}.$$

By using this fact and an approximation argument, we can see that for a subsequence $(k_1, k_2, \ldots, k_n)$ the norm of

$$\begin{aligned} \delta(e_{k_1}e_{k_2}\cdots e_{k_n}) &= \delta(e_{k_1})e_{k_2}\cdots e_{k_n} \\ &+ e_{k_1}\delta(e_{k_2})e_{k_3}\cdots e_{k_n} \\ &+ \cdots \\ &+ e_{k_1}\cdots e_{k_{n-1}}\delta(e_{k_n}) \end{aligned}$$

exceeds $2n\epsilon\sqrt{1/n(1-1/n)}(1-1/n)^{n-1} \approx 2\epsilon\sqrt{n}e^{-1}$, since the products almost become *tensor products* in the above equality. Here we use the fact that A is simple. This is a contradiction for a large n.

Thus we have shown that $\|\delta|C \cap A_n'\| \to 0$. By passing to a subsequence of (A_n) we may suppose that

$$\sum_n \|\delta|C \cap A_n'\| < \infty.$$

Denoting by G_n the unitary group of $C \cap A_n \cap A_{n-1}'$ with $A_0 = 0$, we consider the following integral with respect to normalized Haar measures:

$$\begin{aligned} ih_n &= \int_{G_1 \times \cdots \times G_n} \delta(g_1^* g_2^* \cdots g_n^*) g_n \cdots g_1 \\ &= \int_{G_1} \delta(g_1^*) g_1 + \int_{G_1 \times G_2} g_1^* \delta(g_2^*) g_2 g_1 + \cdots \\ &\quad + \int_{G_1 \times \cdots \times G_n} g_1^* \cdots g_{n-1}^* \delta(g_n^*) g_n g_{n-1} \cdots g_1, \end{aligned}$$

which we can see converges as $n \to \infty$. Since $[ih_n, g] = \delta(g)$ for any unitary g in $G_1 \times \cdots \times G_n$, it follows that $\delta|C \cap A_n = \operatorname{ad} ih_n|C \cap A_n$, which completes the proof.

In the above proof of (1)⇒(2) we did not really use the fact that δ_α is a generator; so we have:

Remark 3.3 If δ is a closed *-derivation in an AF C^*-algebra A and $\mathcal{D}(\delta)$ contains a canonical AF masa of A, then δ is a generator and $\mathcal{D}(\delta)$ is AF.

Proposition 3.4 *Let A be a non type I simple AF C^*-algebra. Then there exists a one-parameter automorphism group α of A such that $\mathcal{D}(\delta_\alpha)$ is AF but does not contain a canonical AF masa of A.*

If A is a UHF algebra of type p^∞ for some $p > 1$, examples for $\mathbf{T} = \mathbf{R}/\mathbf{Z}$ from [12] will be the desired ones.

To deal with a simple AF C^*-algebra A let us proceed as follows. First express A as $\overline{\cup_n A_n}$ with A_n finite-dimensional, as in the proof of 2.1. With the notation there, we assume this time that all the multiplicities $\chi_n(i,j) \geq 4$. We define a homomorphism $\varphi_{n,ij}$ of $A_{nj} \otimes C(\mathbf{T})$ into $A_{nj} \otimes M_{\chi_n(i,j)} \otimes C(\mathbf{T})$ by

$$\varphi_{n,ij}(x)(z) = \begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & \overline{z} \\ 1 & 0 \end{pmatrix} \oplus \oplus_{\ell=0}^{\chi_n(i,j)-5} x(1),$$

and define accordingly $\varphi_n : A_n \otimes C(\mathbf{T}) \to A_{n+1} \otimes C(\mathbf{T})$. Then it follows [11] that the inductive limit C^*-algebra of $(A_n \otimes C(\mathbf{T}), \varphi_n)$ is isomorphic to the original A; we have thus expressed A as $\overline{\cup_n B_n}$ where $B_n = A_n \otimes C(\mathbf{T}) \subset B_{n+1}$.

We define a sequence (H_n) with self-adjoint $H_n \in A_n \otimes 1 \subset B_n$ by $H_1 = 0$ and $H_n = H_{n-1} + \sum_i \sum_j h_{n,ij}$, where $h_{n,ij} \in 1 \otimes M_{\chi_{n-1}(i,j)} \otimes 1 \subset B_n$ is given by

$$h_{n,ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \oplus \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0.$$

Then we define a one-parameter automorphism group α of A by $\alpha_t|B_n = \mathrm{Ad}\, e^{itH_n}|B_n$. Note that $\mathrm{Sp}(H_n) \subset \mathbf{Z}$ and $\alpha_{2\pi} = \mathrm{id}$. Then we can easily conclude that the fixed point algebra A^α is not AF; $K_1(A^\alpha)$ is not trivial. What we need is this property to conclude that $\mathcal{D}(\delta_\alpha)$ does not contain a canonical AF masa of A. Before proving this as a lemma below we shall show that $\mathcal{D}(\delta_\alpha)$ is AF.

Let z be the canonical unitary in $C(\mathbf{T})$ and let $z_n = 1 \otimes z \in A_n \otimes C(\mathbf{T}) = B_n$. We have to use an estimate in the approximation of z_n by a unitary with finite spectrum in B_m for $m > n$. Let u be the image of z_n in B_m. Then the part of $u(z)$ which is not constant in $z = e^{2\pi t},\ t \in [0,1)$, has an equal number of eigenvalues $\exp(\pm i2\pi 2^{n-m}(t+k))$ with $k = 0, 1, \ldots, 2^{m-n}-1$. By using this we approximate u by a unitary $v \in B_m \cap (A_n \otimes 1)'$ with finite spectrum with the order $\|u - v\| \approx 2^{n-m}$ (see [2]). But the norm of $\delta_\alpha|B_m \cap (A_n \otimes 1)'$ can be estimated as $m - n$, which yields $\|\delta_\alpha(u) - \delta_\alpha(v)\| \leq (m-n)\|u - v\|$. Thus we can conclude that we can make the approximation in $\|\cdot\|_{\delta_\alpha}$, which shows that $\mathcal{D}(\delta_\alpha)$ is AF.

Lemma 3.5 *If α is a periodic one-parameter automorphism group of a simple AF C^*-algebra A and A^α is not AF, then $\mathcal{D}(\delta_\alpha)$ does not contain a canonical AF masa of A.*

Proof. Let $\delta = \delta_\alpha$ and suppose that $\mathcal{D}(\delta)$ contains a canonical AF masa C. Then by 3.1 we have a self-adjoint $h \in A$ and an increasing sequence (A_n) of finite-dimensional C^*-subalgebras of A with dense union such that $\delta + \mathrm{ad}\, ih$ leaves A_n invariant and $C \cap A_n \cap A_{n-1}'$ is a masa of $A_n \cap A_{n-1}'$. Also $\delta + \mathrm{ad}\, ih$ vanishes on C. Let β be the one-parameter automorphism group generated by $\delta + \mathrm{ad}\, ih$. Then there is an α-cocycle u such that $\beta_t = \mathrm{Ad}\, u_t\, \alpha_t$. If $\alpha_1 = \mathrm{id}$, then it follows that $\beta_1 = \mathrm{Ad}\, u_1$, i.e., $u_1 \in C$. Since C is AF , we find a self-adjoint $k \in C$ such that $e^{ik} = u_1$. Since $(\delta + \mathrm{ad}\, ih)(k) = 0$, one can conclude that $\delta + \mathrm{ad}(ih - ik)$ generates a one-parameter automorphism group γ with $\gamma_1 = \mathrm{id}$ such that γ leaves each A_n invariant. Since α and γ can be considered as actions of $\mathbf{T}$ and γ is a cocycle-perturbation of α, it follows that the crossed products $A \times_\alpha \mathbf{T}$ and $A \times_\gamma \mathbf{T}$ are isomorphic. Since $A \times_\gamma \mathbf{T}$ is AF as the inductive limit of $A_n \otimes C_0(\mathbf{Z})$ and A^α is a hereditary C^*-subalgebra of $A \times_\alpha \mathbf{T}$, A^α must be AF. This contradiction shows that $\mathcal{D}(\delta)$ cannot contain a canonical AF masa.

If α is not periodic in the proof of 3.4, we could still use the following property for δ_α:

Condition For any $\epsilon > 0$ there exists a $\nu > 0$ with the following property: If $u \in \mathcal{D}(\delta_\alpha)$ is a unitary with $\|\delta_\alpha(u)\| < \nu$ there is a continuous path (u_t) of unitaries in $\mathcal{D}(\delta_\alpha)$ such that $u_0 = 1,\ u_1 = u$, and $\|\delta_\alpha(u_t)\| < \epsilon,\ t \in [0,1]$.

Proposition 3.6 *Let α be a one-parameter automorphism group of a unital simple AF C^*-algebra. If $\mathcal{D}(\delta_\alpha)$ contains a canonical AF masa, then the above Condition for δ_α is satisfied.*

Proof. First suppose that A is finite-dimensional. Then there is an $h = h^* \in A$ such that $\delta_\alpha = \mathrm{ad}\, ih$. The condition $\|\delta_\alpha(u)\| < \nu$ reads

$$\|h - uhu^*\| < \nu.$$

Then the *Condition* follows from Theorem 4.1, which will be given later. Note that here the choice of ν does not depend on A nor δ_α.

Let (A_n) be an increasing sequence of finite-dimensional C^*-subalgebras of A with dense union such that $\alpha_t(A_n) = A_n$. Let $u \in \mathcal{D}(\delta_\alpha)$ be a unitary with $\|\delta_\alpha(u)\| < \nu$. Since $\cup_n A_n$ is dense in $\mathcal{D}(\delta_\alpha)$, there is a sequence (u_n) in $\cup_n A_n$ such that $\|u - u_n\|_{\delta_\alpha} \to 0$. Since $u_n u^* \approx 1$ and $\delta_\alpha(u_n u^*) \approx 0$, we can find a continuous path $(u_n(t))$ in $\mathcal{D}(\delta_\alpha)$ such that $u_n(0) = u_n,\ u_n(1) = u$, and $\|\delta_\alpha(u_n(t))\|$ is of the order of $\|\delta_\alpha(u)\|$. Thus we can suppose that $u \in \cup_n A_n$ and the assertion follows from the previous paragraph.

If $\delta_\alpha = \delta_\beta + \mathrm{ad}\, ih$ with $\beta_t(A_n) = A_n$, there is a sequence (h_n) with $h_n = h_n^* \in A_n$ such that $\|h - h_n\| \to 0$. Then $\delta_\beta + \mathrm{ad}\, ih_n = \delta_\alpha + \mathrm{ad}(ih_n - ih)$ generates an AF locally representable action. Thus we may as well assume that α is AF locally representable. This completes the proof.

If α is periodic and $K_1(A^\alpha)$ is not trivial, then the *Condition* is not satisfied. But we note:

Remark 3.7 In the above proposition the converse does not hold. In fact the example in the proof of 2.1 satisfies the above *Condition*.

By using Proposition 3.6 we can give more examples with the property that that $\mathcal{D}(\delta_\alpha)$ contains no canonical AF masa. For example, as in the proof of Proposition 3.4, suppose that we express A as $\overline{\cup_n B_n}$ with $B_n = A_n \otimes C(\mathbf{T})$ and that we define an α by defining $h_{n,ij}$. This time we choose $h_{n,ij}$ to be of the form:

$$h_{n,ij} = \begin{pmatrix} a_n & 0 \\ 0 & a_n \end{pmatrix} \oplus \begin{pmatrix} -a_n & 0 \\ 0 & -a_n \end{pmatrix} \oplus 0 \oplus \cdots \oplus 0,$$

where (a_n) is an arbitrary sequence such that $a = \inf a_n > 0$. If we had a continuous path (u_t) of unitaries in B_n such that $u_0 = 1,\ u_1 = z_1$, and $\|\delta_\alpha(u_t)\| < a$ for the canonical unitary z_1, we could reach a contradiction as follows. Let $H_n = \sum_j \lambda_j E_j$ be the spectral decomposition with $\lambda_1 > \lambda_2 > \cdots$. By the assumption we have that $\lambda_1 - \lambda_2 \geq a$. Since $\|[H_n, u_t]\| < a$, we can estimate

$$\| \sum_j (\lambda_1 - \lambda_j) E_1 u_t E_j \| < a,$$

which shows that

$$\|E_1 u_t (1 - E_1)\| < 1.$$

Since $E_1 u_t E_1 u_t^* E_1 + \|E_1 u_t (1 - E_1)\|^2 E_1 \geq E_1$, it follows that $E_1 u_t E_1$ is invertible. Since $E_1 u_0 E_1 = E_1$ and $E_1 u_1 E_1 = z_1 E_1$ is a unitary with non-trivial K_1, this is a contradiction. If we have a continuous path of unitaries in $\mathcal{D}(\delta_\alpha)$ with the above property, we approximate the path by a path in $\cup_n B_n$ to reach the contradiction. Since $\cup_n B_n$ is dense in the Banach *-algebra $\mathcal{D}(\delta_\alpha)$, this is possible. Thus we have shown that $\mathcal{D}(\delta_\alpha)$ contains no canonical AF masa. If $\limsup a_n < \infty$, one can also show that $\mathcal{D}(\delta_\alpha)$ is AF.

There is a standard way to construct a one-parameter automorphism group α of a certain UHF algebra through an interaction of a quantum spin system [8]. If the interaction is *quantum*, we expect that any inner perturbation of α is not AF locally representable. We also expect that the quasi-free one-parameter automorphism group of the CAR algebra induced by a one-particle Hamiltonian with continuous spectrum [8, 17] or any inner perturbation of it is not AF locally representable. We conclude this section by posing:

Problem Prove the above conjecture.

4 A homotopy lemma

We prove here a technical lemma which is used in the proof of Proposition 3.6. With an additional assumption on h below (saying the norm is less than 1), this follows from Lemma 5.1 of [6]. To remove this assumption we have to replace a certain approximation argument used there by a *constructive* argument, which will constitute the main part of the proof.

Theorem 4.1 *For any $\epsilon > 0$ there exists a $\nu > 0$ satisfying the following condition: For any unital AF algebra A and $u, h \in A$ such that $u^*u = uu^* = 1$, $h^* = h$, and $\|[h, u]\| < \nu$, there is a rectifiable path $(u_t)_{t\in[0,1]}$ in the unitary group of A such that $u_0 = 1$, $u_1 = u$, $\|[h, u_t]\| < \epsilon$, and the length of (u_t) is smaller than $3\pi + \epsilon$.*

Proof. We may assume that A is finite-dimensional; in particular we assume that h is diagonal.

Let $\delta > 0$ be a sufficiently small number, which will be chosen later depending on ϵ.

Let f be a C^∞-function on $\mathbf{R}$ such that $f \geq 0$, $\int f(t)dt = 1$, and $\operatorname{supp} \hat{f} \subset (-\delta, \delta)$. Define

$$x = \int f(t)e^{ith}ue^{-ith}dt.$$

Then it follows that $\|x\| \leq 1$, $\|[h, x]\| \leq \|[h, u]\|$, and

$$\|x - u\| \leq \|\int f(t)(\operatorname{Ad} e^{ith}(u) - u)dt\| \leq \int f(t)|t|dt\|[h, u]\|,$$

where we have used that

$$\|\operatorname{Ad} e^{ith}(u) - u\| = \|\int_0^t e^{ish}[ih, u]e^{-ish}ds\| \leq |t|\|[h, u]\|.$$

If we denote by E_h the spectral measure of h, then we have that for $x^\# = x$ or x^* and $t \in \mathbf{R}$,

$$E_h(-\infty, t)x^\# E_h[t + \delta, \infty) = 0.$$

We define a projection e_k for each $k \in \mathbf{Z}$ by

$$e_k = E_h[2k\delta, 2(k + 1)\delta).$$

Then there are only a finite number of non-zero e_k. It follows that $\sum_k e_k = 1$ and

$$xe_k x^* \leq E_h[(2k-1)\delta, (2k+3)\delta).$$

We suppose that $\|x - u\| < \mu$, where μ can be made arbitrarily small by choosing ν small. Since $0 \leq 1 - x^*x < 2\mu$ and $0 \leq xe_k x^* - (xe_k x^*)^2 \leq 2\mu xe_k x^*$, we have that $0 \leq 1 - \sum_k xe_k x^* \leq 2\mu$, and $\mathrm{Sp}(xe_k x^*) \subset \{0\} \cup [1-2\mu, 1]$. If x were a unitary (and so $xe_k x^*$ were a projection), we could skip most of the arguments below. What we will do next is to construct a unitary v by using x such that v is close to u and satisfies that $ve_k v^* \leq E_h[(2k-1)\delta, (2k+3)\delta)$.

By multiplying $F_j = E_h[(2j-1)\delta, \infty)$ with $1 - \sum_k xe_k x^*$ whose norm is less than 2μ, we get that

$$\|\sum_{k \geq j} xe_k x^* + F_j xe_{j-1} x^* - F_j\| < 2\mu,$$

which implies that

$$\|F_j xe_{j-1} x^* - xe_{j-1} x^* F_j\| < 4\mu.$$

Since

$$\|(F_j xe_{j-1} x^* F_j)^2 - F_j xe_{j-1} x^* F_j\| < 4\mu + \|F_j((xe_{j-1}x^*)^2 - xe_{j-1}x^*)F_j\| < 6\mu,$$

$F_j xe_{j-1} x^* F_j$ is close to a projection for a small μ. If we denote by f_{j-1}^+ the support projection of $F_j xe_{j-1} x^* F_j$, then we have that

$$\|f_{j-1}^+ - E_h[(2j-1)\delta, \infty) xe_{j-1} x^* E_h[(2j-1)\delta, \infty)\| < 6\mu',$$

where $\mu' = (1 - \sqrt{1 - 24\mu})/12 \approx \mu$, which we again denote by μ below. Note that $\|f_{j-1}^+ - f_{j-1}^+ xe_{j-1} x^*\| < 10\mu$ and that $f_{j-1}^+ \leq E_h[(2j-1)\delta, (2j+1)\delta)$. In the same way we denote by f_j^- the support projection of

$$\begin{aligned}&E_h(-\infty, (2j+1)\delta) xe_j x^* E_h(-\infty, (2j+1)\delta)\\ &= E_h[(2j-1)\delta, (2j+1)\delta) xe_j x^* E_h[(2j-1)\delta, (2j+1)\delta);\end{aligned}$$

then we have that

$$\|f_j^- - E_h[(2j-1)\delta, (2j+1)\delta) xe_j x^* E_h[(2j-1)\delta, (2j+1)\delta)\| < 6\mu.$$

Let $f_j = f_j^- + f_j^+$. Then summing up the above calculations, we obtain that

$$\begin{aligned}\|f_j - xe_j x^*\| &= \|f_j - (E_h[(2j-1)\delta, (2j+1)) + E_h[(2j+1)\delta, (2j+3)\delta)) xe_j x^*\|\\ &< 8\mu + \|f_j - E_h[(2j-1)\delta, (2j+1)\delta) xe_j x^* E_h[(2j-1)\delta, (2j+1)\delta)\\ &\quad - E_h[(2j+1)\delta, (2j+3)\delta) xe_j x^* E_h[(2j+1)\delta, (2j+3)\delta)\|\\ &< 14\mu.\end{aligned}$$

Hence if μ is small, $f_j xe_j (e_j x^* f_j xe_j)^{-1/2}$ defines a partial isometry with initial projection e_j and final projection f_j.

Let $g_j^- = E_h[(2j-1)\delta, (2j+1)\delta) - f_{j-1}^+$, $g_j^+ = E_h[(2j+1)\delta, (2j+3)\delta) - f_{j+1}^-$, and $g_j = g_j^- + g_j^+$. Since $\|g_j x e_{j-1} x^*\| = \|g_j^- x e_{j-1} x^*\| = \|(1-f_{j-1}^+) E_h[(2j-1)\delta, (2j+1)\delta) x e_{j-1} x^*\| < 4\mu$ etc. we obtain that

$$\|g_j x e_j x^* - g_j\| < 10\mu.$$

Since

$$\begin{aligned}\|f_{j-1} x e_j x^*\| &< 6\mu + \|E_h[(2j-1)\delta, (2j+1)\delta) x e_{j-1} x^* E_h[(2j-1)\delta, (2j+1)\delta) x e_j x^*\| \\ &< 10\mu + \|E_h[(2j-1)\delta, (2j+1)\delta) x e_{j-1} x^* x e_j x^*\| \\ &< 12\mu,\end{aligned}$$

we have that

$$\begin{aligned}\|x e_j x^* - g_j\| &\le \|g_j x e_j x^* - g_j\| + \|f_{j-1}^+ x e_j x^*\| + \|f_{j+1}^- x e_j x^*\| \\ &< 34\mu.\end{aligned}$$

Let

$$v = \sum_j f_{2j} x e_{2j} (e_{2j} x^* f_{2j} x e_{2j})^{-1/2} + \sum_j g_{2j-1} x e_{2j-1} (e_{2j-1} x^* g_{2j-1} x e_{2j-1})^{-1/2},$$

which is the unitary part of the polar decomposition of $y = \sum_j f_{2j} x e_{2j} + \sum_j g_{2j-1} x e_{2j-1}$. Since $0 \le 1 - yy^* < 14\mu$, we have that $\|v - y\| < 1/\sqrt{1-14\mu} - 1$. Note also that $v e_j v^* \le E_h[(2j-1)\delta, (2j+3)\delta)$. Since $\|\sum (f_{2j}-1) x e_{2j}\|^2 = \sup_j \|(f_{2j}-1) x e_{2j} x^* (f_{2j}-1)\| \le \sup_j \|x e_{2j} x^* - f_{2j}\|$, we get that

$$\|\sum (f_{2j}-1) x e_{2j}\| < \sqrt{14\mu}.$$

Since $\|(g_{2j-1}-1) x e_{2j-1} x^* (g_{2j-1}-1)\| < 34\mu$, we get that

$$\|\sum (g_{2j-1}-1) x e_{2j-1}\| < \sqrt{34\mu}.$$

Since $\|y - x\| \le \|\sum (f_{2j}-1) x e_{2j}\| + \|\sum (g_{2j-1}-1) x e_{2j-1}\|$, we get $\|y-x\| < \sqrt{14\mu} + \sqrt{34\mu} < 10\sqrt{\mu}$. Hence we get that if μ is sufficiently small,

$$\|v - u\| < \|v - y\| + \|y - x\| + \|x - u\| < 1/\sqrt{1-14\mu} - 1 + 10\sqrt{\mu} + \mu < 10(\mu + \sqrt{\mu}).$$

We assume that the constant $10(\mu + \sqrt{\mu})$ is sufficiently small.

Let $k = \sum_j 2j\delta E_h[2j\delta, 2(j+1)\delta) = \sum 2j\delta e_j$. Then $\|h - k\| < 2\delta$ and $\|[k,u]\| \le 2\|h-k\| + \|[h,u]\| < 4\delta + \|[h,u]\| < 4\delta + \nu$. Since $v e_k v^* \le e_{k-1} + e_k + e_{k+1}$, we have that $k - 2\delta \le vkv^* \le k + 2\delta$. Hence it follows that $\|vkv^* - k\| \le 2\delta$. Since $\|[vu^*, k]\| \le \|[v,k]\| + \|[u,k]\| < 6\delta + \nu$ and $vu^* = e^{ia}$ with $a^* = a \approx 0$, we get that $\|[a,k]\| \approx 0$ (up to the order of $6\delta + \nu$). We take a continuous path $t \in [0,1] \mapsto w_t = e^{ita} u$ of length $\|a\|$. Then since $[k, w_t] = [k, e^{ita}]u + e^{ita}[k,u] \approx 0$ (up to the order of $10\delta + 2\nu$)

and $w_1 = v$, we may replace u by v. From now on we can proceed as in the proof of Lemma 5.1 of [6].

Let $E_n = \sum_{j \geq n} e_j$. Then k equals $2\delta \sum_{n>m} E_n + 2m\delta$, where m is the biggest integer satisfying $E_m = 1$, and the sequence $(E_n)_{n \geq m}$ of projections decreases from 1 to 0 as n increases. Let $F_n = vE_n v^*$. Then we have that $E_{n+1} \leq F_n \leq E_{n-1}$. Since $F_{2n+2} \leq F_{2n+1} \leq F_{2n}$ and $F_{2n+2} \leq E_{2n+1} \leq F_{2n}$, we find a continuous path (w_t) of unitaries of length at most π such that $w_0 = 1$, $[w_t, F_{2n} - F_{2n+2}] = 0$, and $w_1(F_{2n+1} - F_{2n+2})w_1^* = E_{2n+1} - F_{2n+2}$ for all n. Since $\|vkv^* - 4\delta \sum_{2n>m} F_{2n} - 2m\delta\| \leq 2\delta$, we have that $\|w_t vkv^* w_t^* - vkv^*\| \leq 4\delta$ and hence $\|[w_t v, k]\| \leq 6\delta$. Next we find a continuous path (z_t) of unitaries of length at most π such that $z_0 = 1$, $[z_t, E_{2n-1} - E_{2n+1}] = 0$, and $z_1(F_{2n} - E_{2n+1})z_1^* = E_{2n} - E_{2n+1}$. Since $w_1 vkv^* w_1^* = 2\delta(\sum_{2n>m} F_{2n} + \sum_{2n+1>m} E_{2n+1}) + 2m\delta$ and $\|w_1 vkv^* w_1^* - 4\delta \sum_{2n+1>m} E_{2n+1} - 2m\delta\| \leq 2\delta$, we get that $\|z_t w_1 vkv^* w_1^* z_t - w_1 vkv^* w_1^*\| \leq 4\delta$ and hence $\|[z_t w_1 v, k]\| \leq 10\delta$. Since $z_1 w_1 vkv^* w_1^* z_1^* = k$, we can find a continuous path of unitaries from $z_1 w_1 v$ to 1 in the commutant of k, whose length is at most π. (Here we use the fact that the unitary group of eAe for any projection $e \in A$ is connected.) Note that the path obtained by combining these three paths has length at most 3π.

The above calculations show that we can choose δ just depending on ϵ. (For example δ should be smaller than $\epsilon/15$ and much smaller than 1.) Then we choose ν independently (such that ν is smaller than $\epsilon/30$ and $10(\mu + \sqrt{\mu})$ is much smaller than ϵ, where μ is proportional to ν as shown at the beginning of the proof). This concludes the proof.

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